Fuzzy $\mathcal{H}_\infty$ Synchronization for Chaotic Systems with Time-Varying Delay

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In this paper, we propose a new $\mathcal{H}_\infty$ synchronization method for fuzzy model based chaotic systems with external disturbance and time-varying delay. Based on Lyapunov-Krasovskii theory, Takagi-Sugeno (TS) fuzzy model, and linear matrix inequality (LMI) approach, the $\mathcal{H}_\infty$ synchronization controller is presented to not only guarantee stable synchronization but also reduce the effect of external disturbance to an $\mathcal{H}_\infty$ norm constraint. The proposed controller can be obtained by solving a convex optimization problem represented by the LMI. A simulation study is presented to demonstrate the validity of the proposed approach.

Key words: $\mathcal{H}_\infty$ Synchronization; Chaotic Systems; Takagi-Sugeno (TS) Fuzzy Model; Lyapunov-Krasovskii Theory; Time-Varying Delay.

1. Introduction

During the last two decades, synchronization in chaotic dynamic systems has received a great deal of interest among scientists from various research fields since Pecora and Carroll [1] introduced a method to synchronize two identical chaotic systems with different initial conditions. It has been widely explored in a variety of fields including physical, chemical, and ecological systems [2]. In the literature, various synchronization schemes, such as variable structure control [3], Ott-Grebogi-Yorke (OGY) method [4], parameters adaptive control [5], observer-based control [6], active control [7, 8], time-delay feedback approach [9], backstepping design technique [10], complete synchronization [11], and so on, have been successfully applied to the chaos synchronization.

Time-delay often appears in many physical systems such as aircraft, chemical, and biological systems. Unlike ordinary differential equations, time-delayed systems are infinite dimensional in nature and time-delay is, in many cases, a source of instability. The stability issue and the performance of time-delayed systems are, therefore, both of theoretical and practical importance. Since Mackey and Glass [12] first found chaos in a time-delay system, there has been increasing interest in time-delay chaotic systems [13, 14]. The problem of synchronization in time-delayed chaotic systems has also been investigated by several researchers [15 – 21]. In [15], a simple adaptive controller was constructed for synchronizing time-delayed first-order chaotic systems with unknown parameters. Guaranteed cost controllers for time-delayed chaotic systems were proposed to achieve an adequate level of performance in [16, 17]. Also, a delayed feedback controller for stabilizing unstable fixed points of time-delayed chaotic systems was proposed in [18]. The authors in [19] studied a neural network based synchronization method for a class of unknown time-delayed chaotic systems. Recently, impulsive synchronization methods for time-delayed chaotic systems were constructed in [20, 21]. Despite these advances in synchronization for time-delayed chaotic systems, the above research results were restricted to time-delayed chaotic systems without external disturbance. In this paper, we propose a new synchronization method for time-delayed chaotic systems with external disturbance.

In recent years, fuzzy logic has received much attention as a powerful tool for the nonlinear control. Among various kinds of fuzzy methods, Takagi-Sugeno (TS) fuzzy model provides a successful method to describe certain complex nonlinear systems using some local linear subsystems [22, 23]. These linear subsystems are smoothly blended together through fuzzy membership functions. The TS fuzzy model can express a highly nonlinear functional relation with a small number of implications of rules [22]. The TS fuzzy control theory has been applied to chaos control and synchronization in some literatures [24–26]. Recently, the TS fuzzy model based approach to synchronization for time-delayed chaotic systems was proposed in [27].

In real physical systems, one is faced with model uncertainties and a lack of statistical information on the signals. This had led in recent years to an interest in mini-max control, with the belief that $\mathcal{H}_\infty$ control is more robust and less sensitive to disturbance variances and model uncertainties [28]. In order to reduce the effect of the disturbance, Hou et al. [29] firstly adopted the $\mathcal{H}_\infty$ control concept [28] for the chaotic synchronization problem of a class of chaotic systems. In [30], a dynamic controller for the $\mathcal{H}_\infty$ synchronization was proposed. Recently, linear and nonlinear controllers for the $\mathcal{H}_\infty$ anti-synchronization were proposed in [31]. These works [29–31] were all restricted to chaotic systems without time-delay. But in real situation, time-delay is inevitable in the operation of chaotic systems and may deteriorate the synchronization performance. Thus, the synchronization problem for chaotic systems with time-delay was investigated by several researchers [12–21]. To the best of our knowledge, however, for the $\mathcal{H}_\infty$ synchronization of chaotic systems with external disturbance and time-varying delay, there is no result in the literature so far, which still remains open and challenging.

In this paper, a new $\mathcal{H}_\infty$ synchronization method based on the TS fuzzy model is proposed for chaotic systems with external disturbance and time-varying delay. By the proposed scheme, the closed-loop error system is asymptotically synchronized and the $\mathcal{H}_\infty$ norm from the external disturbance to the synchronization error is reduced to a disturbance attenuation level. In contrast to existing synchronization methods [15–21] for time-delayed chaotic systems, an advantage of the proposed method is that a lot of conventional linear controller design methods based on both classical and modern control theory can be easily employed in designing the nonlinear TS fuzzy controllers as well. With the outstanding approximation ability of the TS fuzzy system, the external disturbance in chaotic systems with time-varying delay can be attenuated efficiently in the $\mathcal{H}_\infty$ framework. Based on the Lyapunov-Krasovskii method and the linear matrix inequality (LMI) approach, an existence criterion for the proposed controller is represented in terms of the LMI. The LMI problem can be solved efficiently by using recently developed convex optimization algorithms [32].

This paper is organized as follows. In Section 2, we formulate the problem. In Section 3, an LMI problem for the TS fuzzy model based $\mathcal{H}_\infty$ synchronization of chaotic systems with time-varying delay is proposed. In Section 4, an application example for time-delayed Lorenz system is given, and finally, conclusions are presented in Section 5.

2. Problem Formulation

In system analysis and design, it is important to select an appropriate model representing a real system. As an expression model of a real plant, we use the fuzzy implications and the fuzzy reasoning method suggested by Takagi and Sugeno [22]. Consider a class of time-varying delayed chaotic systems described by

fuzzy rule i: IF ω_1 is $\mu_{i1},\ldots,\omega_s$ is μ_{is} THEN

$$\dot{\boldsymbol{x}}(t) = A_i\boldsymbol{x}(t) + \bar{A}_i\boldsymbol{x}(t-\tau(t)) + \hat{\boldsymbol{\eta}}_i(t), \tag{1}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t), \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector, $\boldsymbol{y}(t) \in \mathbb{R}^m$ is the output vector, $A_i \in \mathbb{R}^{n\times n}$, $\bar{A}_i \in \mathbb{R}^{n\times n}$, and $C \in \mathbb{R}^{m\times n}$ are known constant matrices, $\boldsymbol{\eta}_i(t) \in \mathbb{R}^n$ denotes a bias term which is generated by the fuzzy modelling procedure, ω_j ($j = 1,\ldots,s$) is the premise variable, μ_{ij} ($i = 1,\ldots,r, j = 1,\ldots,s$) is the fuzzy set that is characterized by membership function, r is the number of the IF-THEN rules, and s is the number of the premise variables. $\tau(t)$ is the time-varying delay satisfying

$$0 \le \tau(t) \le \Phi \tag{3}$$

and

$$\dot{\tau}(t) \le \Psi, \tag{4}$$

where Φ and Ψ are scalar constants.

Using a standard fuzzy inference method (using a singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier), the system (1)–(2) is inferred as follows:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\omega)\left[A_i\boldsymbol{x}(t) + \bar{A}_i\boldsymbol{x}(t-\tau(t)) + \boldsymbol{\eta}_i(t)\right], \tag{5}$$

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t), \tag{6}$$

where $\omega = [\omega_1, \ldots, \omega_s]$, $h_i(\omega) = w_i(\omega)/\sum_{i=1}^{r} w_j(\omega)$, $w_i : \mathbb{R}^s \to [0,1]$ $(i = 1, \ldots, r)$ is the membership function of the system with respect to the fuzzy rule i. h_i can be regarded as the normalized weight of each IF-THEN rule and it satisfies

$$h_i(\omega) \geq 0, \quad \sum_{i=1}^{r} h_i(\omega) = 1. \tag{7}$$

The system (5) – (6) is considered as a drive system.

The synchronization problem of system (5) – (6) is considered by using the drive-response configuration. According to the drive-response concept, the controlled fuzzy response system is described by the following rules:

fuzzy rule i: IF ω_1 is $\mu_{i1}, \ldots, \omega_s$ is μ_{is} THEN

$$\dot{\hat{\boldsymbol{x}}}(t) = A_i\hat{\boldsymbol{x}}(t) + \bar{A}_i\hat{\boldsymbol{x}}(t-\tau(t)) + \boldsymbol{\eta}_i(t) + \boldsymbol{u}(t) + G_i\boldsymbol{d}(t), \tag{8}$$

$$\hat{\boldsymbol{y}}(t) = C\hat{\boldsymbol{x}}(t), \tag{9}$$

where $\hat{\boldsymbol{x}}(t) \in \mathbb{R}^n$ is the state vector of the response system, $\hat{\boldsymbol{y}}(t) \in \mathbb{R}^m$ is the output vector of the response system, $\boldsymbol{u}(t) \in \mathbb{R}^n$ is the control input, $\boldsymbol{d}(t) \in \mathbb{R}^k$ is the external disturbance, and $G_i \in \mathbb{R}^{n\times k}$ is a known constant matrix. The fuzzy response system can be inferred as

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}(t) = \sum_{i=1}^{r} h_i(\omega)\big[&A_i\hat{\boldsymbol{x}}(t) + \bar{A}_i\hat{\boldsymbol{x}}(t-\tau(t)) \\ &+ \boldsymbol{\eta}_i(t) + \boldsymbol{u}(t) + G_i\boldsymbol{d}(t)\big],\end{aligned} \tag{10}$$

$$\hat{\boldsymbol{y}}(t) = C\hat{\boldsymbol{x}}(t). \tag{11}$$

Define the synchronization error $\boldsymbol{e}(t) = \hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)$. Then we obtain the synchronization error system

$$\begin{aligned}\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r} h_i(\omega)\big[&A_i\boldsymbol{e}(t) + \bar{A}_i\boldsymbol{e}(t-\tau(t)) \\ &+ \boldsymbol{u}(t) + G_i\boldsymbol{d}(t)\big].\end{aligned} \tag{12}$$

Definition 1. *(Asymptotical synchronization) The error system (12) is asymptotically synchronized if the synchronization error $\boldsymbol{e}(t)$ satisfies*

$$\lim_{t\to\infty} \boldsymbol{e}(t) = 0. \tag{13}$$

Definition 2. *($\mathcal{H}_\infty$ synchronization) The error system (12) is $\mathcal{H}_\infty$ synchronized if the synchronization error $\boldsymbol{e}(t)$ satisfies*

$$\int_0^\infty \boldsymbol{e}^T(t)S\boldsymbol{e}(t)\mathrm{d}t < \gamma^2 \int_0^\infty \boldsymbol{d}^T(t)\boldsymbol{d}(t)\mathrm{d}t, \tag{14}$$

for a given level $\gamma > 0$ under zero initial condition, where S is a positive symmetric matrix. The parameter γ is called the $\mathcal{H}_\infty$ norm bound or the disturbance attenuation level.

Remark 1. *The $\mathcal{H}_\infty$ norm [28] is defined as*

$$\|T_{\mathrm{ed}}\|_\infty = \frac{\sqrt{\int_0^\infty \boldsymbol{e}^T(t)S\boldsymbol{e}(t)\mathrm{d}t}}{\sqrt{\int_0^\infty \boldsymbol{d}^T(t)\boldsymbol{d}(t)\mathrm{d}t}}$$

where T_{ed} is a transfer function matrix from $\boldsymbol{d}(t)$ to $\boldsymbol{e}(t)$. For a given level $\gamma > 0$, $\|T_{\mathrm{ed}}\|_\infty < \gamma$ can be restated in the equivalent form (14). If we define

$$H(t) = \frac{\int_0^t \boldsymbol{e}^T(\sigma)S\boldsymbol{e}(\sigma)\mathrm{d}\sigma}{\int_0^t \boldsymbol{d}^T(\sigma)\boldsymbol{d}(\sigma)\mathrm{d}\sigma}, \tag{15}$$

the relation (14) can be represented by

$$H(\infty) < \gamma^2. \tag{16}$$

In Section 4, through the plot of $H(t)$ versus time, the relation (16) is verified.

The purpose of this paper is to design the controller $\boldsymbol{u}(t)$ guaranteeing the $\mathcal{H}_\infty$ synchronization if there exists the external disturbance $\boldsymbol{d}(t)$. In addition, this controller $\boldsymbol{u}(t)$ will be shown to guarantee the asymptotical synchronization when the external disturbance $\boldsymbol{d}(t)$ disappears.

3. Main Results

In this section, the LMI problem for achieving the $\mathcal{H}_\infty$ synchronization of chaotic systems based on the TS fuzzy model with time-varying delay is presented in the following theorem.

Theorem 1. *For given $\gamma > 0$ and $S = S^T > 0$, if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, and M_j such that*

$$\begin{bmatrix} A_i^T P + PA_i + M_jC + C^T M_j^T + \Phi Q + R & P\bar{A}_i & 0 & PG_i & I \\ \bar{A}_i^T P & -(1-\Psi)R & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\Phi}Q & 0 & 0 \\ G_i^T P & 0 & 0 & -\gamma^2 I & 0 \\ I & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0 \tag{17}$$

for $i, j = 1, 2, \ldots, r$, the controller for the $\mathcal{H}_\infty$ synchronization with the disturbance attenuation level γ is given by

$$u(t) = \sum_{i=1}^{r} h_i(\omega) P^{-1} M_j(\hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t)). \quad (18)$$

Proof. The $\mathcal{H}_\infty$ synchronization controller can be constructed via the parallel distributed compensation. The controller is described by the following rules:

fuzzy rule j: IF ω_1 is $\mu_{j1}, \ldots, \omega_s$ is μ_{js} THEN

$$\boldsymbol{u}(t) = K_j(\hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t)), \quad (19)$$

where $K_j \in \mathbb{R}^{n\times m}$ is the gain matrix of the controller for the fuzzy rule j. The fuzzy controller can be inferred as

$$\begin{aligned} \boldsymbol{u}(t) &= \sum_{j=1}^{r} h_j(\omega) K_j(\hat{\boldsymbol{y}}(t) - \boldsymbol{y}(t)) \\ &= \textstyle\sum_{j=1}^{r} h_j(\omega) K_j C \boldsymbol{e}(t). \end{aligned} \quad (20)$$

The closed-loop error system with the control input (20) can be written as

$$\begin{aligned} \dot{\boldsymbol{e}}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} & h_i(\omega)h_j(\omega)[(A_i + K_j C)\boldsymbol{e}(t) \\ & + \bar{A}_i \boldsymbol{e}(t-\tau(t)) + G_i \boldsymbol{d}(t)]. \end{aligned} \quad (21)$$

Consider the following Lyapunov-Krasovskii functional:

$$V(\boldsymbol{e}(t)) = V_1(\boldsymbol{e}(t)) + V_2(\boldsymbol{e}(t)) + V_3(\boldsymbol{e}(t)), \quad (22)$$

where

$$V_1(\boldsymbol{e}(t)) = \boldsymbol{e}^T(t) P \boldsymbol{e}(t), \quad (23)$$

$$V_2(\boldsymbol{e}(t)) = \int_{-\Phi}^{0}\int_{t+\beta}^{t} \boldsymbol{e}^T(\alpha) Q \boldsymbol{e}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta, \quad (24)$$

$$V_3(\boldsymbol{e}(t)) = \int_{t-\tau(t)}^{t} \boldsymbol{e}^T(\sigma) R \boldsymbol{e}(\sigma)\mathrm{d}\sigma. \quad (25)$$

The time derivative of $V_1(\boldsymbol{e}(t))$ along the trajectory of (21) is

$$\begin{aligned} &\dot{V}_1(\boldsymbol{e}(t)) = \dot{\boldsymbol{e}}(t)^T P \boldsymbol{e}(t) + \boldsymbol{e}^T(t) P \dot{\boldsymbol{e}}(t) = \\ &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\Big\{\boldsymbol{e}^T(t)\big[A_i^T P + PA_i \\ &+ PK_jC + C^T K_j^T P\big]\boldsymbol{e}(t) + \boldsymbol{e}^T(t) P\bar{A}_i \boldsymbol{e}(t-\tau(t)) \\ &+ \boldsymbol{e}^T(t-\tau(t))\bar{A}_i^T P\boldsymbol{e}(t) + \boldsymbol{e}^T(t) PG_i\boldsymbol{d}(t) + \boldsymbol{d}^T(t) G_i^T P\boldsymbol{e}(t)\Big\}. \end{aligned}$$

If we use the inequality $X^TY + Y^TX \le X^T\Lambda X + Y^T\Lambda^{-1}Y$, which is valid for any matrices $X \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times m}$, $\Lambda = \Lambda^T > 0$, $\Lambda \in \mathbb{R}^{n\times n}$, we have

$$\begin{aligned} &\boldsymbol{e}(t)^T PG_i\boldsymbol{d}(t) + \boldsymbol{d}^T(t) G_i^T P\boldsymbol{e}(t) \\ &\le \gamma^2 \boldsymbol{d}^T(t)\boldsymbol{d}(t) + \frac{1}{\gamma^2}\boldsymbol{e}(t)^T PG_iG_i^T P\boldsymbol{e}(t). \end{aligned} \quad (26)$$

Using (26), we obtain

$$\begin{aligned} \dot{V}_1(\boldsymbol{e}(t)) \le \sum_{i=1}^{r}\sum_{j=1}^{r} & h_i(\omega)h_j(\omega)\Big\{\boldsymbol{e}^T(t)\Big[A_i^T P + PA_i \\ & + PK_jC + C^T K_j^T P + \frac{1}{\gamma^2} PG_iG_i^T P\Big]\boldsymbol{e}(t) \\ & + \boldsymbol{e}^T(t) P\bar{A}_i\boldsymbol{e}(t-\tau(t)) + \boldsymbol{e}^T(t-\tau(t))\bar{A}_i^T P\boldsymbol{e}(t) \\ & + \gamma^2\boldsymbol{d}^T(t)\boldsymbol{d}(t)\Big\}. \end{aligned}$$

The time derivative of $V_2(\boldsymbol{e}(t))$ is

$$\dot{V}_2(\boldsymbol{e}(t)) = \Phi\boldsymbol{e}^T(t) Q\boldsymbol{e}(t) - \int_{t-\Phi}^{t} \boldsymbol{e}^T(\sigma) Q\boldsymbol{e}(\sigma)\mathrm{d}\sigma. \quad (27)$$

Using the inequality [33]

$$\begin{aligned} &\left[\int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma\right]^T Q\left[\int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma\right] \\ &\le \Phi\int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)^T Q\boldsymbol{e}(\sigma)\mathrm{d}\sigma, \end{aligned} \quad (28)$$

we have

$$\begin{aligned} \dot{V}_2(\boldsymbol{e}(t)) \le\ & \Phi\boldsymbol{e}^T(t) Q\boldsymbol{e}(t) \\ & - \frac{1}{\Phi}\left[\int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma\right]^T Q\left[\int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma\right]. \end{aligned} \quad (29)$$

Since the time derivative of $V_3(\boldsymbol{e}(t))$ is written as

$$\begin{aligned} &\dot{V}_3(\boldsymbol{e}(t)) = \\ &\boldsymbol{e}(t)^T R\boldsymbol{e}(t) - (1-\dot{\tau}(t))\boldsymbol{e}^T(t-\tau(t)) R\boldsymbol{e}(t-\tau(t)) \\ &\le \boldsymbol{e}(t)^T R\boldsymbol{e}(t) - (1-\Psi)\boldsymbol{e}^T(t-\tau(t)) R\boldsymbol{e}(t-\tau(t)), \end{aligned} \quad (30)$$

we have the derivative of $V(\boldsymbol{e}(t))$ as

$$\dot{V}(\boldsymbol{e}(t)) = \dot{V}_1(\boldsymbol{e}(t)) + \dot{V}_2(\boldsymbol{e}(t)) + \dot{V}_3(\boldsymbol{e}(t))$$

$$\leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau(t)) \\ \int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma \end{bmatrix}^T \cdot \begin{bmatrix} (1,1) & P\bar{A}_i & 0 \\ \bar{A}_i^T P & -(1-\Psi)R & 0 \\ 0 & 0 & -\frac{1}{\Phi}Q \end{bmatrix}\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}(t-\tau(t)) \\ \int_{t-\Phi}^{t}\boldsymbol{e}(\sigma)\mathrm{d}\sigma \end{bmatrix} -\boldsymbol{e}^T(t)S\boldsymbol{e}(t)+\gamma^2\boldsymbol{d}^T(t)\boldsymbol{d}(t), \quad (31)$$

where

$$(1,1) = A_i^T P + PA_i + PK_jC + C^T K_j^T P + \frac{1}{\gamma^2}PG_iG_i^T P + \Phi Q + R + S. \quad (32)$$

If the following matrix inequality is satisfied:

$$\begin{bmatrix} (1,1) & P\bar{A}_i & 0 \\ \bar{A}_i^T P & -(1-\Psi)R & 0 \\ 0 & 0 & -\frac{1}{\Phi}Q \end{bmatrix} < 0 \quad (33)$$

for $i,j = 1,2,\ldots,r$, we have

$$\dot{V}(\boldsymbol{e}(t)) < \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\{-\boldsymbol{e}^T(t)S\boldsymbol{e}(t) + \gamma^2\boldsymbol{d}^T(t)\boldsymbol{d}(t)\} = -\boldsymbol{e}^T(t)S\boldsymbol{e}(t)+\gamma^2\boldsymbol{d}^T(t)\boldsymbol{d}(t). \quad (34)$$

Integrating both sides of (34) from 0 to ∞ gives

$$V(\boldsymbol{e}(\infty)) - V(\boldsymbol{e}(0)) < -\int_0^\infty \boldsymbol{e}^T(t)S\boldsymbol{e}(t)\mathrm{d}t + \gamma^2\int_0^\infty \boldsymbol{d}^T(t)\boldsymbol{d}(t)\mathrm{d}t.$$

Since $V(\boldsymbol{e}(\infty)) \geq 0$ and $V(\boldsymbol{e}(0)) = 0$, we have the relation (14). From Schur's complement, the matrix inequality (33) is equivalent to

$$\begin{bmatrix} A_i^T P + PA_i + PK_jC + C^T K_j^T P + \Phi Q + R & P\bar{A}_i & 0 & PG_i & I \\ \bar{A}_i^T P & -(1-\Psi)R & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\Phi}Q & 0 & 0 \\ G_i^T P & 0 & 0 & -\gamma^2 I & 0 \\ I & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0. \quad (35)$$

If we let $M_j = PK_j$, (35) is equivalently changed into the LMI (17). Then the gain matrix of the control input $\boldsymbol{u}(t)$ is given by $K_j = P^{-1}M_j$. This completes the proof. □

Corollary 1. *Without the external disturbance, if we use the control input $\boldsymbol{u}(t)$ proposed in Theorem 1, the asymptotical synchronization is obtained.*

Proof. When $\boldsymbol{d}(t) = 0$, we obtain

$$\dot{V}(\boldsymbol{e}(t)) < -\boldsymbol{e}^T(t)S\boldsymbol{e}(t) \leq 0 \quad (36)$$

from (34). This guarantees

$$\lim_{t\to\infty}\boldsymbol{e}(t) = 0 \quad (37)$$

from the Lyapunov-Krasovskii theory. This completes the proof. □

Remark 2. *Various efficient convex optimization algorithms can be used to check whether the LMI (17) is feasible. In this paper, in order to solve the LMI, we utilize MATLAB LMI Control Toolbox [34], which implements state-of-the-art interior-point algorithms.*

4. Numerical Example

Consider the following Lorenz system [35] with time-varying delay:

$$\begin{aligned} \dot{x}_1(t) &= -10x_1(t) + 10x_2(t-\tau(t)), \\ \dot{x}_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t), \\ \dot{x}_3(t) &= x_1(t)x_2(t) - \frac{8}{3}x_3(t-\tau(t)), \end{aligned} \quad (38)$$

where $\tau(t) = 0.3(\sin(t)+1)$. By defining two fuzzy sets, we can obtain the following fuzzy drive system that exactly represents the nonlinear equation of the Lorenz system with time-varying delay under the as-

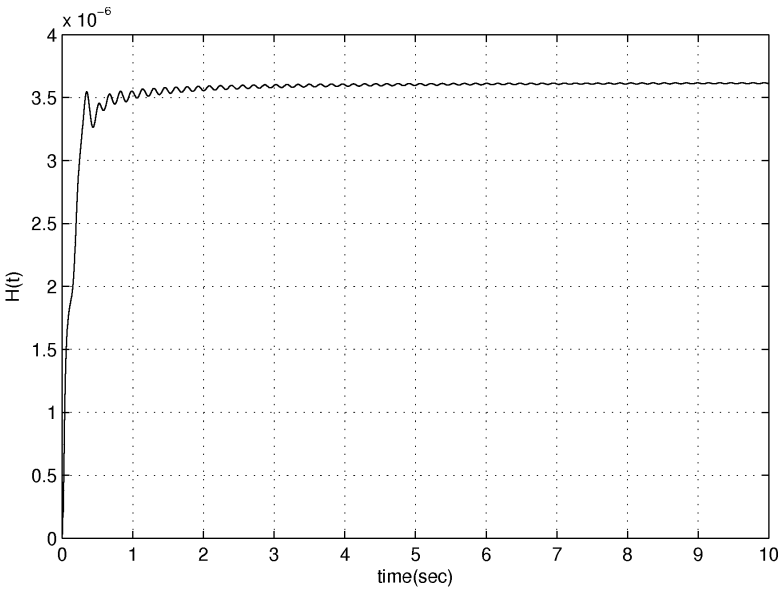

Fig. 1. Plot of $H(t)$ versus time.

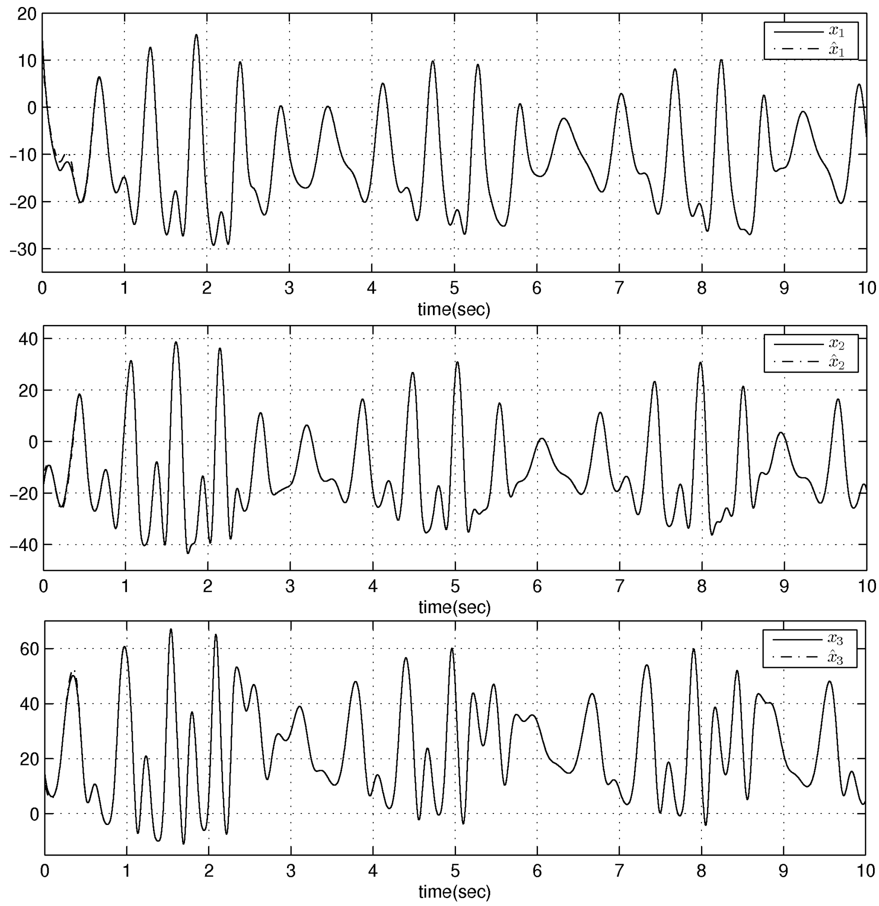

Fig. 2. State trajectories when $\boldsymbol{d}(t)$ is given by (44).

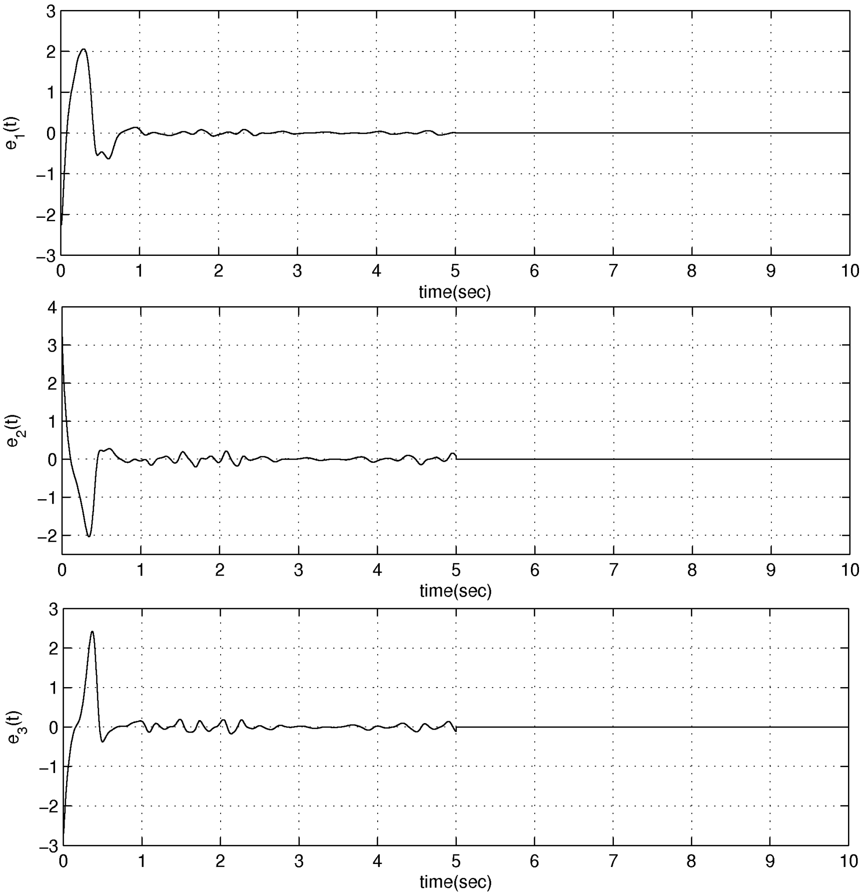

Fig. 3. Synchronization errors when $\boldsymbol{d}(t)$ is given by (44).

sumption that $x_1(t) \in [-d, d]$ with $d = 20$:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{2} h_i(\omega)[A_i x(t) + \bar{A}_i x(t-\tau(t)) + \eta_i], \quad (39)$$

where

$$A_1 = \begin{bmatrix} -10 & 0 & 0 \\ 28 & -1 & -d \\ 0 & d & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -10 & 0 & 0 \\ 28 & -1 & d \\ 0 & -d & 0 \end{bmatrix},$$
$$\bar{A}_1 = \bar{A}_2 = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}, \quad \eta_1 = \eta_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (40)$$

The membership functions are

$$h_1(\omega) = \frac{1}{2}\left(1 + \frac{x_1(t)}{d}\right),$$
$$h_2(\omega) = \frac{1}{2}\left(1 - \frac{x_1(t)}{d}\right). \quad (41)$$

For the numerical simulation, we use the following parameters:

$$\Phi = 0.6, \quad \Psi = 0.3, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$
$$G_1 = G_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}. \quad (42)$$

For the design objective (14), let the $\mathcal{H}_\infty$ performance be specified by $\gamma = 0.2$. Applying Theorem 1 to the fuzzy system (39) yields

$$P = \begin{bmatrix} 4.6667 & -2.6681 & -1.5624 \\ -2.6681 & 5.4757 & -1.9927 \\ -1.5624 & -1.9927 & 4.3585 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} -31.5793 & -62.7630 \\ -151.2358 & -21.3136 \\ 117.5464 & -171.7664 \end{bmatrix},$$

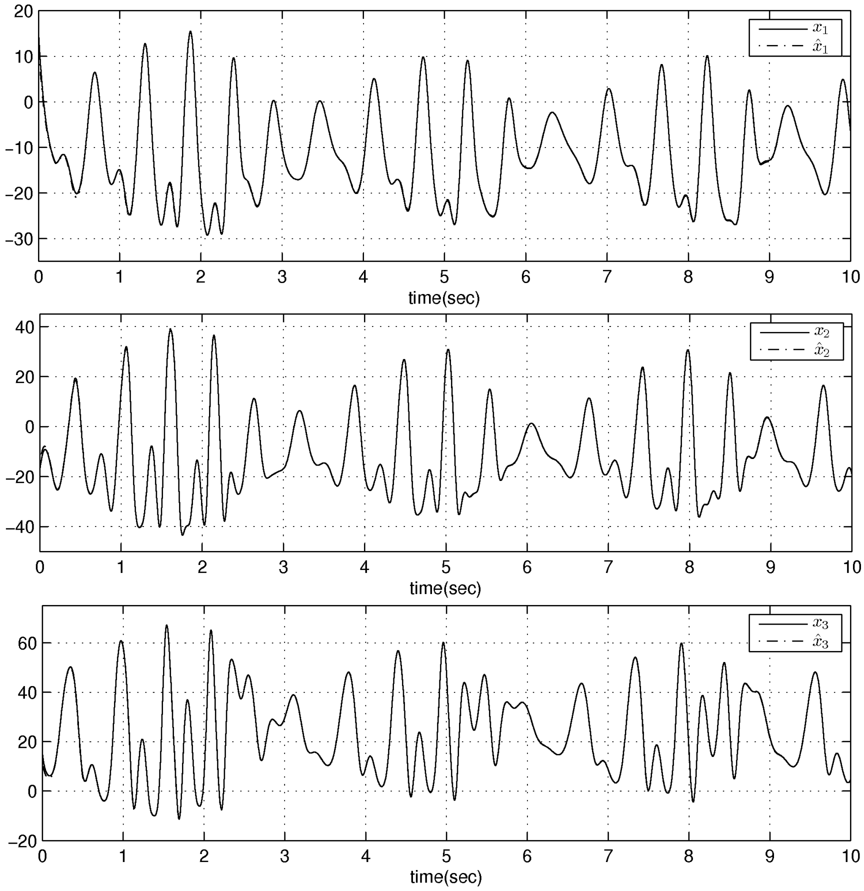

Fig. 4. State trajectories when $\boldsymbol{d}(t)$ is given by (45).

$$M_2 = \begin{bmatrix} -31.5793 & -24.8528 \\ -189.1461 & 16.5966 \\ 79.6362 & -171.7664 \end{bmatrix}.$$

Figure 1 shows the plot of $H(t)$ versus time when $\boldsymbol{d}(t) = \sin(25t)$. Figure 1 verifies $H(\infty) < \gamma^2 = 0.04$. This means that the $\mathcal{H}_\infty$ norm from the external disturbance $\boldsymbol{d}(t)$ to the synchronization error $\boldsymbol{e}(t)$ is reduced within the $\mathcal{H}_\infty$ norm bound γ. Figure 2 shows state trajectories for drive and response systems when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 15.8 \\ -17.48 \\ 15.64 \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \\ \hat{x}_3(0) \end{bmatrix} = \begin{bmatrix} 13.8 \\ -14 \\ 13 \end{bmatrix}, \quad (43)$$

and the external disturbance $\boldsymbol{d}(t)$ is given by

$$\boldsymbol{d}(t) = \begin{cases} w(t), & 0 \le t \le 5, \\ 0, & \text{otherwise}, \end{cases} \quad (44)$$

where $w(t)$ means a Gaussian noise with mean 0 and variance 1. Figure 3 shows that the proposed fuzzy $\mathcal{H}_\infty$ synchronization method reduces the effect of the external disturbance $\boldsymbol{d}(t)$ on the synchronization error $\boldsymbol{e}(t)$. In addition, it is shown that the synchronization error $\boldsymbol{e}(t)$ goes to zero after the external disturbance $\boldsymbol{d}(t)$ disappears.

Next, in order to observe the disturbance attenuation performance for the external disturbance with different frequencies, we change the external disturbance $\boldsymbol{d}(t)$ to

$$\boldsymbol{d}(t) = \begin{bmatrix} \sin(t) \\ \cos(100t) \end{bmatrix} \quad (45)$$

with γ, Φ, Ψ, C, and S remained invariant. In this case, we use the following parameters:

$$G_1 = G_2 = \begin{bmatrix} 0\,1 \\ 1\,0 \\ 0\,1 \end{bmatrix}. \quad (46)$$

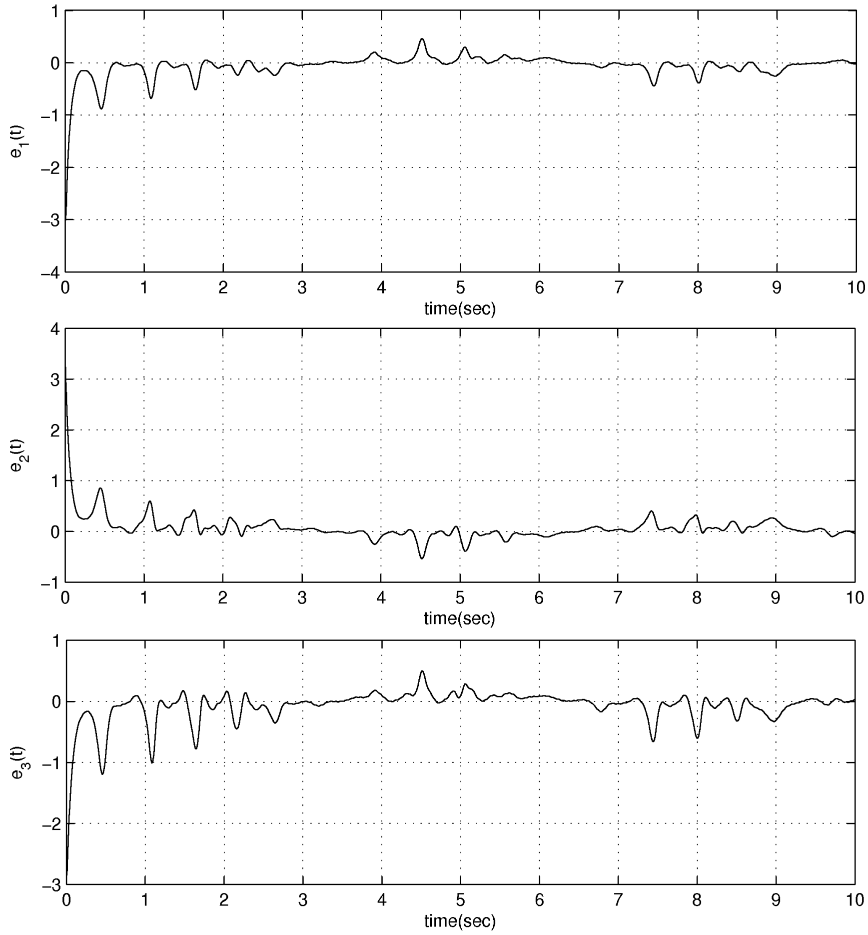

Fig. 5. Synchronization errors when $\boldsymbol{d}(t)$ is given by (45).

Solving for the LMI (17) gives

$$P = \begin{bmatrix} 0.5306 & 0.2730 & -0.4249 \\ 0.2730 & 0.6239 & -0.1357 \\ -0.4249 & -0.1357 & 0.7314 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -110.4396 & -0.7814 \\ -14.8310 & -97.7285 \\ 104.7749 & -105.4104 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} -110.4396 & 51.8094 \\ -67.4217 & -45.1377 \\ 52.1842 & -105.4104 \end{bmatrix}.$$

State trajectories and synchronization error are illustrated in Figure 4 and Figure 5, respectively, when the initial conditions are given by (43). From the simulation results, it can be seen that the resulting disturbance attenuation performance is relatively poor for the disturbance with lower frequency.

5. Conclusion

In this paper, a new $\mathcal{H}_\infty$ design scheme which consists of the fuzzy drive and response systems is proposed for synchronization of chaotic systems with external disturbance and time-varying delay. The TS fuzzy model is used to describe the chaotic drive system with time-varying delay. Based on Lyapunov-Krasovskii theory and LMI approach, the proposed method guarantees the asymptotical synchronization and reduces the $\mathcal{H}_\infty$ norm from the external disturbance to the synchronization error within a disturbance attenuation level. The synchronization for the Lorenz system with time-varying delay is given to illustrate the effectiveness of the proposed scheme. Finally, the proposed scheme has the advantage that it can be effectively used to $\mathcal{H}_\infty$ control and synchronization problems of other nonlinear systems described by a TS fuzzy model with time-varying delay.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1996).
[2] G. Chen and X. Dong, From chaos to order, World Scientific, Singapore 1998.
[3] C. C Wang and J. P. Su, Chaos, Solitons, and Fractal **20**, 967 (2004).
[4] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[5] J. H. Park, Int. J. Nonlinear Sci. Numer. Simul. **6**, 201 (2005).
[6] X. S. Yang and G. Chen, Chaos, Solitons, and Fractals **13**, 1303 (2002).
[7] E. Bai and K. Lonngen, Phys. Rev. E **8**, 51 (1997).
[8] E. W. Bai and K. E. Lonngren, Chaos, Solitons, and Fractals **11**, 1041 (2000).
[9] J. H. Park and O. M. Kwon, Chaos, Solitons, and Fractals **23**, 445 (2005).
[10] J. Hu, S. Chen, and L. Chen, Phys. Lett. A **339**, 455 (2005).
[11] M. Zhan, X. Wang, X. Gong, G. Wei, and C. Lai, Phys. Rev. E **68**, 6208 (2003).
[12] M. C. Mackey and L. Glass, Science **197**, 287 (1977).
[13] J. D. Farmer, Physica D **4**, 366 (1982).
[14] H. Lu, Phys. Lett. A **298**, 109 (2002).
[15] Y. C. Tian and F. Gao, Physica D **117**, 1 (1998).
[16] J. H. Park and O. M. Kwon, Chaos, Solitons, and Fractals **27**, 1011 (2006).
[17] B. Chen, X. Liu, and S. Tong, Chaos, Solitons, and Fractals **34**, 1683 (2007).
[18] X. Guan, G. Feng, C. Chen, and G. Chen, Physica D **227**, 36 (2007).
[19] M. Chen and W. H. Chen, Chaos, Solitons, and Fractals **41**, 2716 (2009).
[20] W. Zhu, D. Xu, and Y. Huang, Chaos, Solitons, and Fractals **35**, 904 (2008).
[21] X. Liu, to be published in Nonlinear Anal.: Theory, Methods Appl. **71**, 1320 (2009).
[22] T. Takagi and M. Sugeno, IEEE Trans. Syst., Man, Cybern **15**, 116 (1985).
[23] K. Tanaka and M. Sugeno, Fuzzy Sets Syst. **45**, 135 (1992).
[24] K. Tanaka, T. Ikeda, and H. O. Wang, IEEE Trans. Circuits Syst. I **45**, 1021 (1998).
[25] K. Y. Lian, C. S. Chiu, T. S. Chiang, and P. Liu, IEEE Trans. Fuzzy Syst. **9**, 539 (2001).
[26] K. Y. Lian, T. S. Chiang, C. S. Chiu, and P. Liu, IEEE Trans. Syst., Man, Cybern. B **31**, 66 (2001).
[27] C. S. Ting, Inform. Sci. **177**, 4314 (2007).
[28] A. Stoorvogel, The $\mathcal{H}_\infty$ Control Problem, Prentice Hall, New York 1992.
[29] Y. Y. Hou, T. L. Liao, and J. J. Yan, Physica A **379**, 81 (2007).
[30] S. M. Lee, D. H. Ji, J. H. Park, and S. C. Won, Phys. Lett. A **372**, 4905 (2008).
[31] C. K. Ahn, Phys. Lett. A **373**, 1729 (2009).
[32] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishinan, Linear matrix inequalities in systems and control theory, SIAM, Philadelphia 1994.
[33] E. Noldus, Int. J. Contr. **41**, 947 (1985).
[34] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, LMI Control Toolbox, Mathworks, Natick 1995.
[35] L. Li, H. Peng, Y. Yang, and X. Wang, Chaos, Solitons, and Fractals **41**, 783 (2009).